CONSERVATION LAWS AND PARABOLIC MONGE-AMPÈRE EQUATIONS.

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1. THE THEOREM

Today I would like to make sense of and describe the ideas behind the proof of the following theorem:

Theorem 1. *An evolutionary scalar parabolic differential equation with at least one non-trivial conservation law is necessarily a parabolic Monge-Ampère equation.*

This talk will be impressionistic, but hopefully I can get the ideas across. Along the way, I will use the following theorem, which I proved in my thesis. Preliminary to stating it, note that given a differential equation, there is an auxilliary differential equation, defined on an infinite dimensional manifold[1], whose solutions are in correspondence with conservation laws to the original equation.

Theorem 2. *The solutions to the auxilliary equation of an evolutionary scalar parabolic differential equation factor through a finite dimensional manifold (in fact, the solutions are defined on the same manifold that the parabolic equation is defined on. More on that later.)*

A corrolary of this statement is that conservation laws for parabolic equations never depend on more than second derivatives of solutions.

This is in stark contrast to other families of differential equations, such as KdV, where the solutions (conservation laws) are necessarily defined on an infinite dimensional manifold, and indeed, there are conservation laws depending on on arbitrarily many derivatives of solutions.

These theorems were introduced by Bryant and Griffiths for parabolics in $1+1$ variables, as well as most of the tools I'll describe below, in the series of papers [1] and [2]. Clelland proved them in 2+1 dimensions in her Thesis [3].

On to the background...

2. PARABOLIC EXTERIOR DIFFERENTIAL EQUATIONS

First, my whole research program is founded on the following idea, which goes back to Cartan: differential equations are (locally) the same data as exterior differential systems. By definition, an exterior differential system is a manifold M and a homogeneous, differentially closed ideal $\mathcal{I}$ in the ring of forms $\Omega^*(M)$ on M. Given a PDE and its corresponding EDS $(M, \mathcal{I})$, the solutions of the PDE correspond to the integral submanifolds of M: those

$$\iota\colon \Sigma \to M$$

so that $\iota^* \mathcal{I} = 0$.

For example, consider a general second order evolutionary parabolic equation for a scalar function u of the variables x^i and t:

$$\frac{\partial u}{\partial t} - F\left(x^i, t, u, \frac{\partial u}{\partial x^i}, \frac{\partial^2 u}{\partial x^i \partial x^j}\right) = 0. \tag{1}$$

[1]The auxilliary equation is defined on the infinite prolongation of the exterior differential system associated to the original equation.

Consider the bundle of 2-jets of functions from $\mathbb{R}^{n+1}$ to $\mathbb{R}$, denoting it by $J^2(\mathbb{R}^{n+1},\mathbb{R})$. The coordinates t, x^i and u define natural jet coordinates

$$p_0, p_i, p_{00}, p_{i0}, p_{ij} = p_{ji}$$

corresponding respectively to the derivatives

$$\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x^i}, \frac{\partial^2 u}{\partial t \partial t}, \frac{\partial^2 u}{\partial t \partial x^i}, \frac{\partial^2 u}{\partial x^i \partial x^j},$$

and we may consider the zero set M of the function

$$p_0 - F\left(x^i, t, u, p_i, p_{ij}\right) = 0$$

on $J^2(\mathbb{R}^{n+1},\mathbb{R})$.

Any section of $J^2(\mathbb{R}^{n+1},\mathbb{R})$ whose image lands in M *would* be a solution, if it were the 2-jet lift of a function u. Fortunately, $J^2(\mathbb{R}^{n+1},\mathbb{R})$ has a natural ideal $\mathcal{I}^+$ that enforce this condition, generated by the following forms:

$$\begin{aligned} \theta_\varnothing &= du - p_i\, dx^i - p_0\, dt \\ \theta_i &= dp_i - p_{ij}\, dx^j - p_{i0}\, dt \\ \theta_0 &= dp_0 - p_{i0}\, dx^i - p_{00}\, dt \end{aligned}$$

More precisely, a section $\sigma\colon \mathbb{R}^{n+1} \to J^2(\mathbb{R}^{n+1},\mathbb{R})$ is locally the 2-jet lift of a function u if and only if $\sigma^* \mathcal{I}^+ = 0$. The pair $(M,\mathcal{I})$, where $\mathcal{I}$ is the restriction of $\mathcal{I}^+$ to M, is the exterior differential system corresponding to the parabolic equation above.

(To answer a common good question that often arises at this point: the differential equation is encoded both in the ideal $\mathcal{I}$ and in the way that M is embedded into $J^2(\mathbb{R}^{n+1},\mathbb{R})$.)

The following definition describes exterior differential systems that locally have an embedding into $J^2(\mathbb{R}^{n+1},\mathbb{R})$ as described above.

Definition 1. A *strongly parabolic system* in $n+1$ variables is a $2n+2+(n+1)(n+2)/2$ dimensional exterior differential system $(M,\mathcal{I})$ such that any point has a neighborhood equipped with a spanning set of 1-forms

(2)
$$\theta_\varnothing, \theta_a, \omega^a, \pi_{ab} = \pi_{ba} \qquad a, b = 0, \dots n$$

that satisfy:

(1) The forms $\theta_\varnothing, \theta_a$ generate $\mathcal{I}$ as a differential ideal.
(2) The structure equations

$$\begin{aligned} d\theta_\varnothing &\equiv \sum_{a=0}^{n} -\theta_a \wedge \omega^a \pmod{\theta_\varnothing} \\ d\theta_0 &\equiv \sum_{a=0}^{n} -\pi_{0a} \wedge \omega^a \pmod{\theta_\varnothing, \theta_b} \qquad b = 0, \dots, n \\ d\theta_i &\equiv \sum_{a=0}^{n} -\pi_{ia} \wedge \omega^a \pmod{\theta_\varnothing, \theta_j} \qquad i, j = 1, \dots, n. \end{aligned}$$

(3) The parabolic symbol relation

$$\sum_{i=1}^{n} \pi_{ii} \equiv \theta_0 \pmod{\theta_\varnothing, \theta_j, \omega^a} \qquad j = 1, \dots, n \qquad a = 0, \dots n.$$

For later use, let $\mathcal{J} = \{\theta_\varnothing, \theta_a, \omega^a\}$.

The extra flexibility in the definition accomodates a study of the geometry of such systems. In particular, the question of what are the invariants up to 'change of coordinates' has a useful answer. After the parabolic symbol itself, the next invariants are the Goursat invariants and the Monge-Ampère invariants.

The Goursat invariants vanish if our parabolic system is evolutionary, which we've assumed, so they vanish automatically for the definition given above (but not for more general parabolic systems.)

3. Parabolic Monge-Ampère Equations

The Monge-Ampère invariants give a geometric characterization of when a parabolic equation is reducible to Monge-Ampère form. This characterization will play into the proof of the theorem, so I state a simplified form here. In the following, $\omega_{(i)}$ is: $(-1)^i$ times the wedge product of each of $\omega^0, \ldots, \omega^n$ excluding ω^i. The Einstein summation convention is applied.

Theorem 3. *A parabolic system $(M, \mathcal{I})$ has a deprolongation to a parabolic Monge-Ampère system if and only if there is a choice of parabolic coframing so that the form*

$$\Upsilon = \theta_i \wedge \omega_{(i)},$$

satisfies the equation

$$d\Upsilon \equiv 0 \quad \left(\textit{mod}\, \theta_\varnothing, \Upsilon, \Lambda^{n+2}\mathcal{J}\right). \tag{3}$$

Later we will see that the form Υ is intimately connected to conservation laws of the parabolic equation. In this context, what the theorem says is that there's a smaller EDS $(M_{-1}, \mathcal{I}_{-1})$ that still captures the structure of solutions. The ideal $\mathcal{I}_{-1}$ is (roughly) generated by $\theta_\varnothing$ and Υ and M_{-1} is a $2n+3$ dimensional manifold.

A second order parabolic differential equation for one function of $n+1$ variables is *Monge-Ampère* if it is quasi-linear in the minor-determinants of the Hessian, so that it can be written in the form

$$p_0 - \sum_{|I|=|J|} A_{I,J}(x^a, u, p_i) H_{I,J} = 0, \tag{4}$$

where the I, J range over subsets of $\{0, \ldots, n\}$ containing 0 and $H_{I,J}$ stands for the minor determinant of the hessian matrix

$$H = \left(\frac{\partial^2 u}{\partial x^a \partial x^b} \right)$$

with rows I and columns J deleted.

An EDSs of the form above corresponds locally to parabolic equations of this Monge-Ampère type.

4. Conservation Laws of Exterior Differential Systems

The following general theory was developed by Bryant and Griffiths in their paper introducing characteristic cohomology.

For a determined system of differential equations (such as a parabolic equation), with n-dimensional solution manifolds, the conservation laws are given by the $n-1$-forms φ for which $d\varphi \in \mathcal{I}$, up to equivalence by trivial such forms: closed $n-1$-forms and ones already in $\mathcal{I}$. The condition that $d\varphi \in \mathcal{I}$ says that φ is closed on all solution manifolds. This motivates the following definition:

Definition 2. Let the characteristic complex be $\bar{\Omega} := \Omega^*(M)/\mathcal{I}$. The characteristic cohomology of $(M, \mathcal{I})$ is the homology of this chain complex. The space of conservation laws is the $n-1$ homology of this complex.

While it turns out that calculating $H(\bar{\Omega})$ involves solving highly nonlinear equations and such, there is a natural filtration of $\Omega^*(M)$, given by

$$F^p = \mathcal{I}^p \wedge \Omega^*(M),$$

and in turn a spectral sequence that converges to $H^*_{dR}(M)$. This spectral sequence makes the characteristic cohomology tractable.

On the E_1 page, we have $H(\bar{\Omega})$ going up the left column, and other spaces to the right. It turns out that the computation of these other spaces comes down to the understanding the symbol of the equation. By a general result of Bryant and Griffiths (though in this application, going back to Vinogradov), everything to the right is zero below the $n-1$ row. This implies immediately that if we restrict attention to a contractible neighborhood,

(1) $H^q(\bar{\Omega}) = 0$ for $q < n-1$ and
(2) there is an exact sequence

$$0 \longrightarrow H^{n-1}(\bar{\Omega}) \longrightarrow E_1^{n-1,1} \xrightarrow{\ \delta\ } E_1^{n-1,2} \longrightarrow \dots$$

Using this result, and a simple computation using the symbol, one quickly concludes that the conservation laws of parabolic $(M, \mathcal{I})$ are in bijection with the kernel δ, and are in fact of the form

$$\Phi \equiv A\Upsilon - \theta_\varnothing \wedge \psi_A \quad \left(\mathrm{mod}\ \theta_\varnothing, \mathcal{I}^2\right), \tag{5}$$

where A satisfies a certain auxilliary equation (which you can write down) and ψ_A defines a linear differential operator from functions A to n-forms. This auxilliary equation is nothing more than the condition which says that one may choose quadratic and higher (in $\mathcal{I}$) terms for Φ so that

$$d\Phi = 0$$

on the nose.

The second theorem actually allows you to strengthen Equation 5 to

$$\Phi \equiv A\Upsilon - \theta_\varnothing \wedge \psi_A \quad \left(\mathrm{mod}\ \theta_\varnothing, \Lambda^{n+2}\mathcal{J}\right),$$

which, while not obvious from what's written here, is a stronger way of saying that the auxilliary equation lives on M, instead of infinite dimensional 'total prolongation' of M. This allows us to make sense of the following: If M is parabolic, and Φ a non-trivial conservation law (so that $A \neq 0$), then

$$0 = d\Phi \equiv A\, d\Upsilon + \cancel{dA \wedge \Upsilon} \quad \left(\mathrm{mod}\ \theta_\varnothing, \Lambda^{n+2}\mathcal{J}, \Upsilon\right)$$

Et voilla!

References

[1] Robert L. Bryant and Phillip A. Griffiths. Characteristic cohomology of differential systems I: General theory. *Journal of the American Mathematical Society*, 8(3):pp. 507–596, 1995.

[2] Robert L. Bryant and Phillip A. Griffiths. Characteristic cohomology of differential systems II: Conservation laws for a class of parabolic equations. *Duke Math. J.*, 78(3):531–676, 06 1995.

[3] Jeanne N. Clelland. Geometry of conservation laws for a class of parabolic partial differential equations. *Selecta Mathematica*, 3(1):1–77, 1997.